A synopsis on

The Study of solutions of certain Diophantine equations

Submitted to

Swami Ramanand Teerth Marathwada University
Nanded-431606, Maharashtra, India

For the Degree of

Doctor of Philosophy(Ph.D.)

In Mathematics
(Faculty of Sciences and Technology)

Submitted by

Ms. Priyanka R.Satpute

Research Student

Under the Guidance of

Dr. Usha K.Sangale

Assistant Professor
School of Mathematical Sciences,
SRTM University, Nanded-431606.

August 2023

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Ms Priyanka R Satpute

1. Abstract

E. Catalan in 1885 stated, without proof, that the only solutions of the equation $y^x - x^y = 1$ in integers (x, y) is $(2, 3)$. However, a proof of this result was already given by Moret-Blanc in 1876. Since then, there has been plenty of work generalizing Catalan's problem. A general version of the type $y^x - x^y = z^2$ was recently considered by Florian Luca and Maurice Mignotte [4]. We propose to study more general equation of the type $y^x - x^y = kz^2$ for various integer values of k. Another related problem of interest is the famous Lebesgue-Ramanujan-Nagell equations. They are diophantine equations of the form $x^2 + D = p^n$, where p is an odd prime, D is squarefree integer. We would like to pursue such type of problems.

2. Introduction

The study of Diophantine equations, that is solutions in integers of equations with integer coefficients, has its origin in the works of Greek mathematicians. Euclid's algorithm tells us that the linear diophantine equation

$$ax + by = c, \quad \text{where } a, b, c \text{ are given integers},$$

is solvable in integers x, y if and only if the $\gcd(a, b)$ divides c. Using elementary ideas of divisibility, it can be shown that the equation

$$x^2 + y^2 = z^2$$

has an infinity of integer solutions. However, the story is completely different if the degree exceeds 2. The famous *Fermat's Last theorem* (FLT), stated by Pierre de Fermat in the year 1637, says that the equation

$$x^n + y^n = z^n, \tag{1}$$

has no solutions in positive integers x, y, z if $n \geq 3$. By elementary arguments, it follows that it is enough to consider only prime values of n. Ernst Kummer's attempts to solve this problem has led to the development of a new subject called *algebraic number theory*. Pursuing the ideas of Kummer, FLT has been proved for all primes upto four million! Unfortunately, in mathematics a statement like FLT is valid only if it is true for *all* integers. One of the biggest achievements of the last century was the work of Andrew Wiles who proved in 1995 that FLT is in fact true! He was awarded the most prestigious *Abel Prize* for this marvellous achievement. The study of FLT tells us that several areas of mathematics are sometimes deeply connected and the interconnections often reveals hidden relationships which are not apparent to the unsuspecting eyes! The connections between modular forms, elliptic curves and FLT are related was the main idea that led

to the final solution. Great mind like Louis Mordell, Gerd Faltings, Goro Shimura, Yutaka Taniyama, Gerhard Frey, Ken Ribet, Jean-Pierre Serre, John Coates, Richard Taylor, Nick Katz, and many others have contributed to this fruitful journey which eventually led to the proof of FLT by Wiles.

There is another interesting diophantine problem that has attracted lot of attention. *Eugéne Charles Catalan* in 1844 conjectured that the only solutions in positive integers of the equation

$$x^a - y^b = 1 \tag{2}$$

for $a, b > 1, x, y > 0$ is $x = 3, a = 2, y = 2, b = 3$.

Alan Baker developed a theory in 1966-67, now known as *Baker's method*, which gives a lower bound for the absolute value of linear combinations of logarithms of algebraic numbers. This powerful tool was instrumental in solving many important classical problems in transcendental number theory. In 1976, Robert Tijdeman applied this method of linear form in logarithms and gave an effective upper bound for the solutions of (2) in terms of a and b. This essentially means that the Catalan conjecture is verified for all but finitely many cases. Finally, in 2002, Preda Mihăilescu completely settled Catalan's conjecture. For a lucid introduction to this topic, one may refer to P. Ribenboim's book [8].

In 1850, V. A. Lebesgue [6] considered a related problem and proved that the equation

$$x^2 + 1 = y^n, \quad x, y, n \in \mathbb{N}, \min\{x, y, n\} > 1 \tag{3}$$

has no solutions (x, y, n). S. Ramanujan in 1913 conjectured that the equation

$$x^2 + 7 = 2^n \tag{4}$$

has solutions only when $n = 3, 4, 5, 7$ and n. This was proved by T. Nagell in 1948 and showed that the corresponding values of x are $x = 1, 3, 5, 11$ and 181 respectively.

The equations of the type

$$x^2 + D = p^n \tag{5}$$

where p is an odd prime are now known as *generalized Lebesgue-Ramanujan-Nagell (LRN)* equations.

There are numerous methods invented to attack diophantine equations. The study of the diophantine equations mentioned above has enriched number theory. It has opened up new vistas for research with plenty of open problems. These problems have shown a beautiful interplay of elementary techniques from classical number theory, concepts from algebraic number theory, diophantine approximations, modular forms, linear forms in logarithms. We wish to pursue some problems of the above mentioned type in our research work.

3. Review of literature

We shall study the various techniques that have been used to attack diophantine equations. We propose to study

1. Elementary methods in number theory. We shall follow the book *Introduction to analytic number theory* by Tom M. Apostol [2].

2. We shall update our knowledge on algebraic number theory by following the books: *Problems in algebraic numbers theory* written by Ram Murty and J. Esmonde [7] and *Introductory algebraic number Theory* written by Saban Alaca and Kenneth Williams [1].

3. For transcendental number theory, we shall refer the book *Transcendental number theory* by A. Baker.

4. For Baker's theory on the linear forms in logarithms we shall refer the book *Exponential diophantine equations* written by T. N. Shorey and R. Tijdeman [9].

5. To understand the relation between diophantine equations and modularilty of elliptic curves, we shall study relevant portions from the book by H. Cohen [5].

6. We shall undertake a serious study of recent published papers on diophantine equations (of the type mentioned in the beginning).

4. Importance of research

a. We hope our research will enrich the existing literature in area of number theory.

b. Our research output may get published in internationally reputed peer reviewed journals.

c. Our work may open up avenues for collaboration with leading mathematicians from across the globe.

d. Our pursuits may give a direction to future researchers (from the department) who are interested in number theory.

e. Our work may give visibility to our department.

f. It will earn me a Ph D degree, which will be the beginning of my career.

5. Objectives of research/proposed hypothesis

1. The main goal is to study the solution of diophantine equations.

2. We shall study the equation $y^x - x^y = z^2$. We hope that this will give us an idea of how to attack the general diophantine equation
$$y^x - x^y = kz^2$$
for various values of k.

3. The hope that the study of FLT will give us an idea of how to solve the generalized FLT
$$ax^n + by^n = cz^n \tag{6}$$
for some values of a, b, c and n.

4. We shall study the equation
$$x^2 + D = p^n \tag{7}$$
and a more general version of this equation.

5. While working on the above problems, it is possible that some other related problems may pop up. We shall pursue them also.

6. Significance of study

The most significant part of research will be to add our bit to the existing literature, thereby enhancing it further. We hope that our study will find its place in well-known research journals. This may induce collaboration from other Institutes. This work will train me to be a good researcher and I hope I can pursue it further with my future students.

7. Plan of research

First Year. We shall undertake a serious study of Elementary and Algebraic number theory. We shall follow the books [7] and [1].

Second Year. We shall learn advanced techniques in number theory. Our aim would be to thoroughly understand Baker's theory of *linear forms in logarithms* following the book [3], class numbers of quadratic fields from the book [1], relation between diophantine equations, modular forms and elliptic curves from the book [5].

Third Year. We shall study the research papers on *generalized Lebesgue-Ramanujan-Nagell (LRN)* equations and try to solve some open problems. We would also consider replacing the number D in (5) with Dy^k and study the solutions for various k and n.

Fourth Year. We shall study the generalized FLT and try to prove results in some special cases.

Fifth Year. We shall undertake the study of the general diophantine equation

$$y^x - x^y = kz^2$$

for various values of k.

Sixth Year. We shall attend workshops and conferences to give talks on our work. The writing of the thesis will be undertaken.

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Ms. Satpute Priyanka Ramesh
Research Student

Dr. Usha K.Sangale
Research Guide